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ASYMPTOTIC ANALYSIS OF GENERALIZED FUNCTIONS THROUGH SOME GENERALIZED INTEGRAL TRANSFORMS AND FRAMES

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Abstract: *Generalized asymptotics is an important research subject that has attracted the attention of many authors in diverse areas. In recent years it has become common to analyze the asymptotic behavior of distributions through the asymptotic behavior of some integral transforms or frames. Some Abelian- and Tauberian-type results from Buralieva et al. (Funct. Anal. Appl. 53:3-10, 2019); Buralieva et al. (Chatzakou et al. (eds) Women in Analysis and PDE, in the series "Research Perspectives Ghent Analysis and PDE Center" Birkhäuser, Cham, 2024); Buralieva (Funct. Anal. Appl. 57(1):29-39, 2023); and Buralieva (Asian-Eur. J. Math., 2025) for the quasiasymptotics of tempered and Lizorkin distributions through the asymptotics of the appropriate integral transforms and frames are presented.*

Keywords: *Quasiasymptotics, Stockwell transform, wavelet transform, directional short-time Fourier transform, frames, Abelian- and Tauberian-type results.*

1 Introduction

The term generalized asymptotics refers to asymptotic analysis of generalized functions or distributions. The most developed approaches are those of Vladimirov, Drozhizhnov and Zavialov [25], and of Kanwal and Estrada [9]. The quasiasymptotic behavior of distributions was introduced by Zavialov [27], and further developed and applied by Drozhzhinov, Vladimirov and Zavialov, in order to analyze the asymptotic behavior of tempered distribution through the asymptotic behavior of Laplace transform. This approach is followed by authors [15] such that they made a great contribution in this field too, and defined other asymptotics, called S-asymptotics. The second approach is somehow related to asymptotic expansions of generalized functions and the Ces'aro behavior.

The asymptotic analysis of a distribution can be done through generalized integral transforms [2, 3, 4, 5, 20] and frames [6, 7, 22] (and references therein), such that one can obtain Abelian- and Tauberian-type results. The Abelian-type result means the result that is obtained for the generalized integral transform or generalized frame coefficients, knowing the asymptotic behavior of the appropriate distribution, while the Tauberian-type result refers to the result that is obtained for the distribution, knowing the asymptotic behavior of the generalized integral transform or generalized frame coefficients of that distribution. A Tauberian-type result does not always exist, and when it does usually requires an additional condition, called Tauberian boundedness.

We present some Abelian- and Tauberian-type results for a special choice of distributions, namely tempered and Lizorkin distributions. Some Abelian-type results for the quasiasymptotic behavior of Lizorkin distributions relating to the asymptotic behavior of Stockwell and wavelet transforms from [3] and [4] are given. Also, one Tauberian-type result at infinity for the Stockwell transform from [3] is presented. Abelian-type result for the quasiasymptotic boundedness of the directional short-time Fourier transform (DSTFT) of tempered distribution, and Abelian- and Tauberian-type result for the DSTFT with fixed direction from [2] are given. The Abelian- and Tauberian-type result from [7] that we present is asymptotic characterization of a tempered distribution through the asymptotic behavior of its localized frame coefficients.

2 Preliminaries

2.1 Notation and spaces

The notation $\langle f, \varphi\rangle$ means dual pairing between a distribution f and a test function φ. When the limit $\lim_{x\to x_0}\frac{f(x)}{g(x)} = 0$ holds for the measurable functions f and g, the notation $f(x) = o(g(x))$, $x \to x_0$ is used. For the inner product of f and g in a Hilbert space we use the notation $(f, g)_{\mathcal{H}}$. The operators for modulation, translation and dilatation are defined by $M_a f(\cdot) = e^{ia\cdot} f(\cdot)$, $T_b f(\cdot) = f(\cdot - b)$ and $D_{\frac{1}{c}} f(\cdot) = |c| f(c\,\cdot)$, $a, b \in \mathbb{R}$, $c \in \mathbb{R}\setminus\{0\}$, respectively, for a measurable function f on $\mathbb{R}$.

$\mathcal{S}(\mathbb{R}^n)$ stands for the well known Schwartz space, i.e. the space of rapidly decreasing smooth functions $\varphi \in C^\infty(\mathbb{R}^n)$ for which all seminorms

$$\rho_{k,n}(\varphi) = \sup_{x\in\mathbb{R}^n, |n|\le k} (1 + |x|^2)^{k/2} |\varphi^{(n)}(x)|,\;\; k \in \mathbb{N}_0 \tag{1}$$

are finite. Its dual space is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ [23]. The space of Lizorkin test functions, i.e. the space of highly time-frequency localized test functions is denoted by $\mathcal{S}_0(\mathbb{R}^n)$. An element $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$ if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^m\varphi(x)dx = 0$ for all $m \in \mathbb{N}_0^n$, (see example [14]). Its dual space $\mathcal{S}_0'(\mathbb{R}^n)$ is the space of Lizorkin distributions. The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is the domain of the DSTFT [21] and DSTFT with fixed direction [1], the space $\mathcal{S}'(\mathbb{R})$ is the domain of the polinomially localied frames [7], while the space of Lizorkin distributions $\mathcal{S}_0'(\mathbb{R})$ is the domain of the Stockwell and wavelet transforms [20] and [14], respectively.

2.2 Quasiasymptotics

The quasiasymptotics is a kind of generalized asymptotics, defined with respect to a slowly varying function. A function L is called slowly varying at the origin (resp. at infinity)(see for example in [15]) if it is a measurable real-valued function, defined and positive on the interval $(0, A]$ (resp. $[A, \infty)$), $A > 0$ and

$$\lim_{\varepsilon\to 0^+}\frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \qquad \left(\text{resp.} \lim_{\lambda\to\infty}\frac{L(a\lambda)}{L(\lambda)} = 1\right) \quad \text{for each} \quad a > 0.$$

The distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ has quasiasymptotic behavior of degree $\alpha \in \mathbb{R}$ at the origin (resp. at infinity) with respect to L if there exists $h \in \mathcal{S}'(\mathbb{R}^n)$ such that for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(x)\right\rangle = \langle h(x), \varphi(x)\rangle \;\left(\text{resp.} \;\lim_{\lambda\to\infty}\left\langle \frac{f(\lambda x)}{\lambda^\alpha L(\lambda)}, \varphi(x)\right\rangle = \langle h(x), \varphi(x)\rangle\right). \tag{2}$$

When (2) holds, we use the notation $f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon)h(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathrm{R}^n)$ (resp. $f(\lambda x) \sim \lambda^\alpha L(\lambda)h(x)$ as $\lambda \to \infty$). Here, it is important to point that the distribution h does not have an arbitrary form. In [9], [15], [25] one can find that such h must be homogeneous with degree of homogeneity α, i.e., $h(ax) = a^\alpha h(x)$, for each $a > 0$.

Let $f_m(x) = x^m f(x)$, $m \in \mathbb{N}$, let $f \in \mathcal{S}'(\mathbb{R})$, and let (2) hold, then

$$f_m(\varepsilon x) \sim \varepsilon^{\alpha+m} L(\varepsilon)h_m(x) \text{ as } \varepsilon \to 0^+ \tag{3}$$

(resp. $f_m(\lambda x) \sim \lambda^{\alpha+m} L(\lambda)h_m(x)$ as $\lambda \to \infty$) in $\mathcal{S}'(\mathbb{R}), m \in \mathbb{N}$, holds, see [4, Lemma 3.1].

These notations admit obvious generalizations; for example, one can consider the quasiasymptotic behavior of distributions in $\mathcal{S}_0'(\mathbb{R})$ (see Section 3). More about quasiasymptotic behavior and its properties one can find in [15].

A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is quasiasymptotically bounded at degree $\alpha \in \mathbb{R}$ at the origin (resp. at infinity) with respect to L if

$$\left|\left\langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(x)\right\rangle\right| = O(1) \ \left(\text{resp. } \left|\left\langle \frac{f(\lambda x)}{\lambda^\alpha L(\lambda)}, \varphi(x)\right\rangle\right| = O(1)\ \lambda \to \infty\right), \tag{4}$$

and we use the notation $f(\varepsilon x) = O(\varepsilon^\alpha L(\varepsilon))$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R}^n)$, (resp. $f(\lambda x) = O(\lambda^\alpha L(\lambda))$ as $\lambda \to \infty$), [26].

3 Asymptotic analysis of Lizorkin distributions

Asymptotic analysis of Lizorkin distribtions is done through several integral transforms [3, 4, 5, 20] and references therein, and some frame coefficients [22]. In this section we present some asymptotic results for the Lizorkin distribution relating to the asymptotic results for the Stockwell and wavelet transforms (see Subsection 3.1 and 3.2, respectively).

3.1 Asymptotic results through Stockwell transform

In 1996, Stockwell et al. [24] defined the Stockwell transform of $f \in L^2(\mathbb{R})$ with respect to $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (such that $\int_{\mathbb{R}} g(x) = 1$), for all $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$ as

$$S_g f(b,a) \ = \ \frac{1}{\sqrt{2\pi}}|a| \int_{\mathbb{R}} e^{-ixa} f(x)\overline{g}(a(x-b))dx.$$

The Stockwell transform of $f \in \mathcal{S}_0'(\mathbb{R})$ with respect to $g \in S_0(\mathbb{R})$, for all $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$ is defined by Saneva et al. [20] as

$$S_g f(b,a) = \frac{1}{\sqrt{2\pi}} \langle f, \overline{M_a T_b D_{\frac{1}{a}} g}\rangle.$$

Even more, the authors [20] have done some asymptotic analysis for the Lizorkin distributions through the generalized Stockwell transform. In the present paper we present an Abelian- and Tauberian-type result for the asymptotic behavior of Lizorkin distributions through the generalized Stockwell transform from [3], and also we give an Abelian-type result for the asymptotic behavior for the Stockwell transform of f and f_m at the origin from [3] and [4], respectively.

Proposition 3.1. *[3, Proposition 3.10] Let $\alpha \in \mathbb{R}$, let $f \in \mathcal{S}_0'(\mathbb{R})$, and let $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$. Assume that, for every $(b,a) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$, the limits*

$$\lim_{\lambda \to \infty} \frac{S_g f\left(\lambda b, \frac{a}{\lambda^2}\right)}{\lambda^\alpha L(\lambda)} \in \mathbb{C}$$

exist and there are $C > 0$, $s > 1, p \in \mathbb{N}$, and $0 < \lambda_0 \le 1$ such that

$$\frac{\left|S_g f(\lambda b, \frac{a}{\lambda^2})\right|}{\lambda^\alpha L(\lambda)} \le C\left(|a| + \frac{1}{|a|}\right)^{-s} (1+|b|)^p,$$

for all $(b,a) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ and $\lambda \ge \lambda_0$. Then there exists a homogeneous distribution h such that the quasiasymptotic behavior (2) as $\lambda \to \infty$ holds.

Remarks 3.2. *Let us notice that if (2) holds, then $S_g f(\lambda b, \frac{a}{\lambda^2}) = o(\lambda^\alpha L(\lambda))$ is the Abelian-type result of Proposition 3.1 (see [3, Proposition 3.2]).*

Proposition 3.3 is Abelian-type result at the origin for the Stockwell transform of Lizorkin distribution f.

Proposition 3.3. *[3, Proposition 3.1] Let $\alpha \in \mathbb{R}$, and let $f \in \mathcal{S}_0'(\mathbb{R})$. Assume that (2) holds as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R})$. Then the Stockwell transform of f with respect to a window $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ satisfies*

$$S_g f(\varepsilon b, \varepsilon a) = o(\varepsilon^\alpha L(\varepsilon)) \text{ as } \varepsilon \to 0^+.$$

Proposition 3.4 is Abelian-type result relating the quasiasymptotic behavior (3) at the origin to the asymptotics of the Stockwell transform of f_m with respect to the variable b.

Proposition 3.4. *[4, Proposition 3.4] Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $f \in \mathcal{S}_0'(\mathbb{R})$. Assume that (3) holds as $\varepsilon \to 0^+$. Then the Stockwell transform of f_m with respect to window $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ satisfies*

$$S_g f_m(\varepsilon b, a) = o(\varepsilon^{\alpha+m} L(\varepsilon)) \text{ as } \varepsilon \to 0^+.$$

3.2 Asymptotic results through wavelet transform

The wavelet transform of a signal $f \in L^2(\mathbb{R})$ is defined with respect to the wavelet $g \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} g(x)dx = 0$, see example [12] on p. 203. In [14] on p. 103, generalized wavelet transform of $f \in \mathcal{S}_0'(\mathbb{R})$ and $g \in \mathcal{S}_0(\mathbb{R})$, is defined by

$$W_g f(b, a) = |a|^{-1/2} \int_{\mathbb{R}} f(x)\overline{g}\left(\frac{x-b}{a}\right) dx \tag{5}$$

for all $b \in \mathbb{R}$ and $a \in \mathbb{R}^+$.

The distributional relation between Stockwell and wavelet transforms for $f \in \mathcal{S}_0'(\mathbb{R})$ and $g \in \mathcal{S}_0(\mathbb{R})$,

$$S_g f(b, a) = \frac{e^{-iba}|a|^{1/2}}{\sqrt{2\pi}} W_{e^{ix}g} f\left(b, \frac{1}{a}\right) \tag{6}$$

(see Proposition 4.2 in [3]) is important for the presented asymptotic results for the wavelet transform in this section. The result in Corollary 3.5, 3.6 and 3.7 is the corresponding Abelian-type result for the wavelet transform having the result in Remark 3.2, Propositions 3.3 and 3.4 for the Stockwell transform, respectively.

Corollary 3.5. *[3, Corollary 4.6 a)] Let $\alpha \in \mathbb{R}$, and let $f \in \mathcal{S}_0'(\mathbb{R})$. Assume that (2) holds as $\lambda \to \infty$. Then the wavelet transform of f with respect to a window $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ satisfies*

$$W_{e^{ix}g} f(\lambda b, \frac{\lambda^2}{a}) = o(\lambda^{\alpha+1} L(\lambda)) \text{ as } \lambda \to \infty.$$

Corollary 3.6. *[3, Corollary 4.5 a)] Let $\alpha \in \mathbb{R}$, and let $f \in \mathcal{S}_0'(\mathbb{R})$. Assume that (2) holds as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R})$. Then the wavelet transform of f with respect to a window $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ satisfies*

$$W_{e^{ix}g} f(\varepsilon b, \frac{1}{\varepsilon a}) = o(\varepsilon^{\alpha-1/2} L(\varepsilon)) \text{ as } \varepsilon \to 0^+.$$

Corollary 3.7. *[4, Corollary 3.3] Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}$, and let $f \in \mathcal{S}_0'(\mathbb{R})$. Assume that (3) holds as $\varepsilon \to 0^+$. Then the wavelet transform of f_m with respect to a window $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ satisfies*

$$W_{e^{ix}g} f_m(\varepsilon b, \frac{1}{a}) = o(\varepsilon^{\alpha+m} L(\varepsilon)) \text{ as } \varepsilon \to 0^+.$$

4 Asymptotic analysis of tempered distributions

Asymptotic analysis of tempered distribtions is done through several integral transforms [2, 7, 19, 25], and frame coefficients [6, 7, 18] and some references therein. In this section we present some asymptotic results for the tempered distributions relating to the asymptotic results for the DSTFT and DSTFT with fixed direction (see Subsection 4.1), and to the coefficients of the polinomially localized frames (see Subsection 4.2).

4.1 Asymptotic results through DSTFT

Grafakos and Sansing [11] first defined a sensitive variant of short-time Fourier transform, called directional short-time Fourier transform, but their definition did not give them a stable reconstruction formula. Some years later Giv [10] provided another definition for the DSTFT which has a stable reconstruction formula. The DSTFT of $f \in L^1(\mathbb{R}^n)$ by Giv is defined as

$$DS_g f(u,b,a) = \int_{\mathbb{R}^n} f(x)\overline{g}_{u,b,a}(x)dx, \tag{7}$$

where $(u,b,a) \in \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$, $g \in \mathcal{S}(\mathbb{R})$, and with respect to the Gabor ridge functions $g_{u,b,a}$: $\mathbb{R}^n \to \mathbb{C}$ defined as

$$g_{u,b,a,}(x) = e^{2\pi i x\cdot a} g(x \cdot u - b).$$

The DSTFT defined by Giv is extended to the space of tempered distributions [21], while in [1] the DSTFT with fixed direction is defined.

Let $u \in \mathbb{S}^{n-1}$ be fixed. The DSTFT of $f \in \mathcal{S}^{'}(\mathbb{R}^n)$ with respect to $g \in \mathcal{S}(\mathbb{R})$ with fixed direction u is defined by

$$DS_{g,u}f(b,a) = \int_{\mathbb{R}^n} f(x)\overline{g(u\cdot x - b)}e^{-2\pi i x\cdot a}dx, \tag{8}$$

where $(b,a) \in \mathbb{R} \times \mathbb{R}^n$, [1].

Proposition 4.1 is Abelian-type result for the quasiasymptotic bounddedness for the DSTFT assuming that tempered distribution is quasiasymptotically bounded at the origin (resp. at infinity).

We use the following notation $g_{\frac{1}{\varepsilon}}(x) = g(\frac{x}{\varepsilon})$.

Proposition 4.1. *[2, Proposition 3.1.] Let $\alpha \in \mathbb{R}$, let $g \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$, and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Assume that (4) holds. Then,*

$$DS_{g_{1/\varepsilon}} f(u, \varepsilon b, a/\varepsilon) = O(\varepsilon^{\alpha+n}L(\varepsilon)) \quad as \quad \varepsilon \to 0^+.$$

Similar Abelian-type result holds for the quasiasymptotic boundedness of DSTFT at infinity. Proposition 4.2 is Abelian- and Tauberian-type result for the asymptotic behavior of DSTFT with fixed directional parameter at the origin.

Proposition 4.2. *[2, Proposition 4.3.] Let $\alpha \in \mathbb{R}$, let $f \in \mathcal{S}'(\mathbb{R}^n)$, let $g \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$, and let $u \in \mathbb{S}^{n-1}$ be a fixed directional parameter. The conditions that*

(i) there exist limits

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha+n}L(\varepsilon)} DS_{g_{\frac{1}{\varepsilon}},u} f\left(\varepsilon b, \frac{a}{\varepsilon}\right) \in \mathbb{C}$$

for every $(b,a) \in \mathbb{R} \times \mathbb{R}^n$ and

(ii) there exist $C > 0$, $s \geq 0$ and $0 < \epsilon_0 \leq 1$ such that

$$\left|\frac{DS_{g_{\frac{1}{\varepsilon}},u} f\left(\varepsilon b, \frac{a}{\varepsilon}\right)}{\varepsilon^{\alpha+n}L(\varepsilon)}\right| \leq C\,(1+|b|)^s\,(1+|a|)^s \tag{9}$$

for all $(b,a) \in \mathbb{R} \times \mathbb{R}^n$ and $0 < \varepsilon \leq \epsilon_0$ are necessary and sufficient conditions for the existence of a homogeneous distribution h such that (2) holds as $\varepsilon \to 0^+$.

Similar Abelian- and Tauberian-type result holds for quasiasymptotics at infinity.

4.2 Asymptotic results through localized frames

In the literature there exist several types of localized frames (by Gröchenig, Balan, Casazza, Heil and Laundau), but we use the one defined by Gröchenig [13]. Before we state the definition of a polinomially localized frames, we will give some basic definitions from the frame theory [8].

A sequence $(u_n)_{n\in I}$ from a Hilbert space $\mathcal{H}$, where I is a countable index set, is called a frame for $\mathcal{H}$ if there exist positive constants A and B such that $A\|h\|^2 \le \sum_{n\in I} |(h,u_n)_{\mathcal{H}}|^2 \le B\|h\|^2$ for every $h \in \mathcal{H}$. For a frame $(u_n)_{n\in I}$ for $\mathcal{H}$, there always exists a frame $(v_n)_{n\in I}$ for $\mathcal{H}$ such that every $h \in \mathcal{H}$ can be reconstructed from the frame coefficients $(h,u_n)_{\mathcal{H}}, n \in I$, by the frame expansion formula $h = \sum_{n\in I} (h,u_n)_{\mathcal{H}} v_n$.

A frame for $\mathcal{H}$ that is at the same time a Schauder basis for $\mathcal{H}$ is called a Riesz basis for $\mathcal{H}$. The unique biorthogonal sequence to a Riesz basis $(r_n)_{n=1}^{\infty}$ for $\mathcal{H}$ is also a Riesz basis for $\mathcal{H}$ and will be denoted by $(\widetilde{r}_n)_{n\in I}$.

According to [13], a frame $(u_n)_{n=1}^{\infty}$ for $\mathcal{H}$ is called polynomially localized with decay $\gamma > 0$ with respect to a Riesz basis $(r_n)_{n=1}^{\infty}$ for $\mathcal{H}$ if there exists a constant C_γ such that

$$\max\{|(u_m, r_n)_{\mathcal{H}}|, |(u_m, \widetilde{r}_n)_{\mathcal{H}}|\} \le C_\gamma (1 + |m-n|)^{-\gamma},\ \forall m, n \in \mathbb{N}.$$

Gröchenig [13] defined the theory of polinomially localized frames to Banach spaces, authors [16, 17] defined this theory for the space of tempered distributions, while authors [6] conducted asymptotic analysis for the tempered distributions through polinomially localized frames.

The frame coefficients for $f \in \mathcal{S}'(\mathbb{R})$ and sequence $(e_n)_{n=1}^{\infty}$ with elements from $\mathcal{S}(\mathbb{R})$ that is a frame for $L^2(\mathbb{R})$, and polinomially localized frame with respect to the Hermite basis $(h_n)_{n=1}^{\infty}$ of $L^2(\mathbb{R})$ (defined as $h_n = \mathrm{h}_{n-1}$, $n \in \mathbb{N}$, where $\mathrm{h}_n(t) = (2^n n! \sqrt{\pi})^{-1/2} (-1)^n e^{t^2/2} \frac{d^n}{dt^n}(e^{-t^2})$, $n \in \mathbb{N}_0$) are

$$\langle f(x), e_n \rangle.$$

Theorem 4.3 is Abelian- and Tauberian-type result for quasiasymptotic behavior at the origin of tempered distributions through polinomially localized frames with respect to the Hermite basis.

Theorem 4.3. *[7, Theorem 6.1] Assume that $(e_n)_{n=1}^{\infty}$ is a sequence with elements from $\mathcal{S}(\mathbb{R})$ that is a frame for $L^2(\mathbb{R})$ and polynomially localized with respect to the Hermite basis $(h_n)_{n=1}^{\infty}$ with decay s for every $s \in \mathbb{N}$. Let L be slowly varying at the origin and $\alpha \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R})$, there exists a tempered distribution g such that (2) holds at the origin if and only if the following two conditions hold:*

(i) the limit $\lim_{\varepsilon\to 0^+} \langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, e_n \rangle$ exists for each $n \in \mathbb{N}$;

(ii) there exist $k \in \mathbb{N}_0$ and $\varepsilon_0 \in (0,\infty)$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{n\in\mathbb{N}} \frac{|\langle f(\varepsilon x), e_n \rangle|}{\varepsilon^\alpha L(\varepsilon) n^k} < \infty.$$

A similar result holds for quasiasymptotics at infinity (see [7, Theorem 6.2]).

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